International Journal of Control

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/tcon20

Cluster consensus for coupled harmonic oscillators under a weighted cooperative-competitive network

Cui-Qin Ma, Tian-Ya Liu & Yun-Bo Zhao

INTERNATIONAL JOURNAL OF CONTROL
2022, VOL. 95, NO. 12, 3344–3352
https://doi.org/10.1080/00207179.2021.1971770

Cluster consensus for coupled harmonic oscillators under a weighted cooperative-competitive network

Cui-Qin Ma[a], Tian-Ya Liu[a] and Yun-Bo Zhao[b]

[a]School of Mathematical Sciences, Qufu Normal University, Qufu, People's Republic of China; [b]Department of Automation, University of Science and Technology of China, Hefei, People's Republic of China

ABSTRACT

Cluster consensus is investigated for multiple coupled harmonic oscillators under a weighted cooperative-competitive network. Consensus protocols for three categories of communication networks are constructed by employing a weighted gain, and sufficient conditions for guaranteeing cluster consensus are obtained. It is found that under the proposed protocols, the states of all oscillators can be guaranteed to reach periodic orbits that are the same in frequency no matter which cluster the oscillators belong to. In particular, cluster partitions here are not given a prior, but are determined by the communication topology among oscillators. Numerical examples are given to validate the effectiveness of theoretical results.

ARTICLE HISTORY
Received 2 March 2021
Accepted 14 August 2021

KEYWORDS
Multi-agent systems; cluster consensus; coupled harmonic oscillators; a weighted cooperative-competitive network

1. Introduction

Central to the distributed coordination of multi-agent systems (MASs), the concept of consensus refers to the phenomenon where agents therein converge to a common value guided by local coordination (Ma et al., 2010; Ma & Zhang, 2010; Olfati-Saber & Murray, 2004; Ren & Beard, 2005; Zhao et al., 2020; Zheng & Wang, 2012). In recent years, it is observed that in many practically meaningful scenarios such as foraging activities with mixed specifies (Dolby & Grubb, 1998), robotic sorting (Prorok et al., 2017), etc., the agents in an MAS may be grouped into multiple clusters and maintain multiple steady states, where consensus can be reached only for the agents within the same cluster, thus the name 'cluster consensus'. There are mainly two types of cluster consensus of interest, that is, communication topology with cooperative interactions (Belykh et al., 2003; Develer & Akar, 2021; Feng et al., 2014; Guo et al., 2020; Qin et al., 2017; Qin & Yu, 2013; Wu et al., 2009; Yu & Wang, 2010; Zhang & Ji, 2018; Zhao et al., 2019) and communication topology with cooperative-competitive interactions (Ge et al., 2018; Liu et al., 2020; Zhan & Li, 2018; Zhao et al., 2020), where all the agents of the former are cooperative while for the latter the agents can be both cooperative and competitive. To reflect this kind of communication interactions, signed digraphs are often used, where cooperative agents within the same cluster are connected by non-negative edge weights, while competitive agents from different clusters are connected by negative edge weights.

Cluster consensus has already attracted much attention. In Yu and Wang (2010), based on the in-degree balance couple assumption, necessary and sufficient algebraic conditions ensuring cluster consensus are given for single-integrator MASs. These conditions are then improved in Qin and Yu (2013) by replacing the hard-to-verify algebraic conditions by sufficient graphic conditions. Under the acyclic partition assumption, cluster consensus for general linear MASs is addressed regardless of coupling strength among agents. In Feng et al. (2014), single-integrator dynamics in Yu and Wang (2010) is extended to the double-integrator model. In Ge et al. (2018), a generalised cluster formation framework is proposed which covers (Qin & Yu, 2013; Yu & Wang, 2010) as its special cases. The aforementioned works are based on two assumptions, i.e. topology graph with acyclic partition and in-degree balance couple condition. To relax these restrictive assumptions, weighted cooperative-competitive graphs are introduced to describe communication topologies in Zhan and Li (2018). Moreover, the topology is divided into three types and corresponding protocols are presented for agents ensuring cluster consensus. Then, in Zhao et al. (2020), a novel communication topology is introduced for cluster consensus, where the concept of structure balance (Altafini, 2013; Ma & Xie, 2020) and its criterion are extended, without the need of augmented undirected graph in Zhan and Li (2018).

Many practical MASs can be modelled as multiple coupled harmonic oscillators (MCHOs). For example, for n objects of equal mass linked by dampers with each object being equipped with identical springs, their dynamics can be described by n coupled harmonic oscillators (Ballard et al., 2010; Ren, 2008; Zhang et al., 2018). In electrical networks, identical LC oscillators coupled through passive impedances can also be represented by coupled harmonic oscillators (Tuna, 2017; Zhang et al., 2018). MCHOs are of great importance since each harmonic oscillator is coupled with its neighbour oscillators and MCHOs have potential applications in multi-agent networks

CONTACT Yun-Bo Zhao ybzhao@ustc.edu.cn

with repetitive movements, such as robots patrol and surveillance (Ballard et al., 2010; Ren, 2008; Zhou et al., 2012). Despite its importance, the cluster consensus of MCHOs has not been fully considered. Indeed, most existing works are dependent upon acyclic partition and balance couple graph assumptions (Su et al., 2013; Zhang & Ji, 2018; Zhao et al., 2019), while for weighted cooperative-competitive networks, existing works consider only single-integrator dynamics (Liu et al., 2020; Zhan & Li, 2018; Zhao et al., 2020).

In this work we consider cluster consensus of MCHOs under weighted cooperative-competitive graphs. The weighted cooperative-competitive graphs are categorised into three types, i.e. interactively balanced, interactively sub-balanced and interactively unbalanced. For each graph type, distributed protocols are proposed. Different from the single-integrator dynamics adopted in Zhan and Li (2018), where the state matrix of the closed-loop system is just the Laplacian matrix, here, the second-order linear harmonic oscillator model makes the state matrix of the closed-loop system coupled with the Laplacian matrix of the graph, and consequently brings difficulties for the convergence analysis of the cluster consensus protocols. To deal with these difficulties, two useful lemmas are introduced in advance and sufficient conditions are given for cluster consensus. It is proven that oscillators finally converge to periodic orbits with the same frequency regardless of clusters that they belong to. In addition, unlike existing studies (Belykh et al., 2003; Feng et al., 2014; Ge et al., 2018; Guo et al., 2020; Liu et al., 2020; Qin et al., 2017; Qin & Yu, 2013; Wu et al., 2009; Yu & Wang, 2010; Zhang & Ji, 2018; Zhao et al., 2019) where the cluster partitions are given a prior, we here do not assume such information, and show that cluster number is dependent on communication topology among oscillators.

The contributions of this work are threefold. First, cluster consensus of MCHOs under generic weighted cooperative-competitive graphs is considered, which is rare to see in existing works. Second, distributed protocols for three categories of communication graphs are proposed, and pinning control is employed to realise cluster consensus for interactively unbalanced graphs. Besides the above three graph categories proposed in Zhan and Li (2018), in this paper, a special case of the network where the augmented undirected graph does not have directed cycles is also considered. Finally, sufficient conditions for ensuring cluster consensus are given.

The remainder of this paper is organised as follows. In Section 2, we review relevant results on graph theory and formulate the problem of interest. In Section 3, we establish the main results under the weighted cooperative-competitive networks. Numerical simulations are presented in Section 4. Finally, the paper is concluded in Section 5.

Notations. $\mathbb{R}^n$ is the set of n-dimensional real column vectors and $\mathbb{R}^{m\times n}$ is the set of $m \times n$ real matrices. I_n is $n \times n$ identity matrix. $\mathbf{1}_n = (1, 1, \ldots, 1)^T$. $\mathbf{0}$ is a matrix (vector) with appropriate dimension. $\mathrm{diag}(a_1, a_2, \ldots, a_n)$ is a diagonal matrix with diagonal elements $a_i, i = 1, 2, \ldots, n$. $\mathrm{Re}(\lambda)$ is the real part of λ. A^T is the transpose of A. For $A = (A_{ij}) \in \mathbb{R}^{n\times n}$ and $B = (B_{ij}) \in \mathbb{R}^{n\times n}$, $A \circ B = (A_{ij}B_{ij})_{ij}$.

2. Preliminaries and problem formulation

2.1 Preliminaries

Generally, a weighted cooperative-competitive network with n agents can be described by a signed digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}, \mathcal{D})$, where $\mathcal{V} = \{1, 2, \ldots, n\}$ is the node set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set, $\mathcal{A} = (a_{ij}) \in \mathbb{R}^{n\times n}$ is the non-negative adjacency matrix, and $\mathcal{D} = (d_{ij}) \in \mathbb{R}^{n\times n}$ is the weighted matrix reflecting the cooperative and competitive relations in a network. $d_{ij} \neq 0 \Leftrightarrow (j, i) \in \mathcal{E}$. In particular, $d_{ij} > 0$ iff the relation between i and j is cooperative, $d_{ij} < 0$ iff the relation between i and j is competitive. For $\mathcal{A} = (a_{ij})$, $a_{ij} \geq 0$ and $a_{ij} > 0 \Longleftrightarrow (j, i) \in \mathcal{E}$. We assume $a_{ii} = 0$, $i \in \mathcal{V}$. $\mathcal{N}_i = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$ is the neighbour set of agent i. $L = (l_{ij})$ is the Laplacian matrix of $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}, \mathcal{D})$, satisfying $l_{ii} = \sum_{j=1,j\neq i}^{n} a_{ij}$, and $l_{ij} = -a_{ij}d_{ij}, i \neq j$.

When relations among all agents are cooperative, i.e. $d_{ij} \geq 0$, we let $d_{ij} = 1$ for $a_{ij} > 0$ and $d_{ij} = 0$ for $a_{ij} = 0$, then $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}, \mathcal{D})$ reduces to non-negative digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, in which Laplacian $L_s = (l_{s,ij}) \in \mathbb{R}^{n\times n}$ is called standard Laplacian with $l_{s,ii} = \sum_{j=1,j\neq i}^{n} a_{ij}$, $l_{s,ij} = -a_{ij}, i \neq j$.

A directed path $\mathcal{P}_{ij}$ in $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}, \mathcal{D})$ is a sequence of edges $(i, \mathcal{V}_1)(\mathcal{V}_1, \mathcal{V}_2) \cdots (\mathcal{V}_k, j)$, where i, $\mathcal{V}_1$, $\mathcal{V}_2$, $\cdots$, $\mathcal{V}_k, j$ are distinct nodes in $\mathcal{V}$. The corresponding weighted product of $\mathcal{P}_{ij}$ is $T_{ij} = d_{\mathcal{V}_1 i} d_{\mathcal{V}_2\mathcal{V}_1} \cdots d_{j\mathcal{V}_k}$. A directed cycle $\mathcal{C}_i$ is a directed path $\mathcal{P}_{ij}$ with $i = j$, and its weighted product is $C_i = d_{\mathcal{V}_1 i} d_{\mathcal{V}_2\mathcal{V}_1} \cdots d_{i\mathcal{V}_k}$.

Let $\hat{\mathcal{G}} = (\mathcal{V}, \hat{\mathcal{E}}, \hat{\mathcal{A}}, \hat{\mathcal{D}})$ be the bidirectional graph of $\mathcal{G}$, where $\hat{\mathcal{A}} = (\hat{a}_{ij}) \in \mathbb{R}^{n\times n}$, $\hat{\mathcal{D}} = (\hat{d}_{ij}) \in \mathbb{R}^{n\times n}$, and

$$\hat{a}_{ij} = \begin{cases} a_{ji}, & (j,i) \notin \mathcal{E}, (i,j) \in \mathcal{E} \\ a_{ij}, & \text{else} \end{cases}$$

$$\hat{d}_{ij} = \begin{cases} 1/d_{ji}, & (j,i) \notin \mathcal{E}, (i,j) \in \mathcal{E} \\ d_{ij}, & \text{else} \end{cases}$$

The digraph $\mathcal{G}$ is interactively balanced if all $C_i = 1$ in $\hat{\mathcal{G}}$. If $\mathcal{G}$ does not contain directed cycles and there is at least one $C_i \neq 1$ in $\hat{\mathcal{G}}$, then $\mathcal{G}$ is interactively sub-balanced. $\mathcal{G}$ is interactively unbalanced if $\mathcal{G}$ contains directed cycles and there exists at least one $C_i \neq 1$ in $\hat{\mathcal{G}}$.

The following two lemmas on algebraic graph theory are useful.

Lemma 2.1 **((Zhan & Li, 2018)):** *There exists a nonsingular matrix $K = \mathrm{diag}(k_{11}, k_{12}, \ldots, k_{1n})$ such that $K^{-1}(\mathcal{A} \circ \mathcal{D})K = \mathcal{A}$, i.e. $K^{-1}LK = L_s$, if $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}, \mathcal{D})$ is interactively balanced and has a spanning tree.*

Lemma 2.2 **((Ren & Beard, 2005)):** *Let L_s be standard Laplacian matrix in cooperative network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$. L_s has a single eigenvalue zero and all other eigenvalues have positive real parts if and only if $\mathcal{G}$ has a spanning tree.*

2.2 Problem formulation

Consider the following coupled harmonic oscillators,

$$\begin{cases} \dot{r}_i(t) = v_i(t), \\ \dot{v}_i(t) = -\alpha r_i(t) + u_i(t), \quad i = 1, 2, \ldots, n, \end{cases} \tag{1}$$

where $r_i(t) \in \mathbb{R}$ represents the position of the ith oscillator, $v_i(t) \in \mathbb{R}$ is the velocity of the ith oscillator, $\alpha > 0$ is the frequency of the oscillators and $u_i(t) \in \mathbb{R}$ is the control input to be designed. We are interested to achieve cluster consensus with the weighted cooperative-competitive networks described by $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}, \mathcal{D})$.

Definition 2.3: For the system in (1), cluster consensus is achieved if there exists distributed control protocol $u_i(t)(i = 1, 2, \dots, n)$ and $c_{ij} \in \mathbb{R}$ such that

$$\lim_{t\to\infty} (r_j(t) - c_{ij} r_i(t)) = 0,$$
$$\lim_{t\to\infty} (v_j(t) - c_{ij} v_i(t)) = 0, \quad i, j = 1, 2, \dots, n,$$

where c_{ij} is determined by the communication relations in $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}, \mathcal{D})$.

Remark 2.1: In accordance with the definition of cluster consensus (Qin & Yu, 2013), oscillators i and j are in the same cluster iff $c_{ij} = 1$. Particularly, if $c_{ij} = 1, \forall i, j$, then all oscillators achieve consensus. Furthermore, oscillators achieve bipartite consensus if $c_{ij} = \pm 1, i, j = 1, \dots, n$. In this sense, consensus and bipartite consensus can be regarded as special cases of cluster consensus.

The following control protocol is proposed.

$$u_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}\left(d_{ij} v_j(t) - v_i(t)\right), \quad i = 1, 2, \dots, n, \tag{2}$$

Remark 2.2: Obviously, protocol (2) is distributed since it is only based on the information of agent i and its neighbours. In particular, if $d_{ij} = 1$ for $(j, i) \in \mathcal{E}$, protocol (2) degenerates to the standard consensus protocol in Ren (2008). If $d_{ij} = \pm 1$ for $(j, i) \in \mathcal{E}$, protocol (2) is reduced to the bipartite consensus protocol in Liu et al. (2018). It is worth noting that d_{ij} in this paper is not limited to ± 1, and hence protocol (2) is more general.

Applying protocol (2) to the system in (1), we have

$$\begin{pmatrix} \dot r(t) \\ \dot v(t) \end{pmatrix} = \begin{pmatrix} \mathbf{0} & I_n \\ -\alpha I_n & -L \end{pmatrix} \begin{pmatrix} r(t) \\ v(t) \end{pmatrix} \triangleq Q \begin{pmatrix} r(t) \\ v(t) \end{pmatrix}, \tag{3}$$

where $r(t) = (r_1(t), r_2(t), \dots, r_n(t))^T$, $v(t) = (v_1(t), v_2(t), \dots, v_n(t))^T$, and L is Laplacian matrix of $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}, \mathcal{D})$.

2.3 Useful lemmas

Before proceeding to the main results, we give the following properties of state matrix Q in (3).

Lemma 2.4: *Suppose $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}, \mathcal{D})$ is interactively balanced and has a spanning tree. Let λ be an eigenvalue of Q, then there exists an eigenvalue ξ of L_s, satisfying $\lambda^2 + \lambda\xi + \alpha = 0$. Furthermore, if φ and η are, respectively, the left and right eigenvectors associated with λ, then, there exists an invertible matrix P such that $\varphi = (P^{-1})^T (v_l^T, -\frac{\lambda}{\alpha} v_l^T)^T$ and $\eta = P(v_r^T, \lambda v_r^T)^T$, where v_l and v_r are the left and right eigenvectors associated with ξ, respectively.*

Proof: Since $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}, \mathcal{D})$ is interactively balanced and has a spanning tree, by Lemma 2.1, there exists an invertible matrix $K = \mathrm{diag}(k_{11}, k_{12}, \dots, k_{1n})$ such that $K^{-1} L K = L_s$. Let $P = \mathrm{diag}(K, K)$, then P is invertible, and

$$P^{-1} Q P = \begin{pmatrix} \mathbf{0} & I_n \\ -\alpha I_n & -K^{-1} L K \end{pmatrix} = \begin{pmatrix} \mathbf{0} & I_n \\ -\alpha I_n & -L_s \end{pmatrix}. \tag{4}$$

Suppose λ is an eigenvalue of Q and η is its right eigenvector, i.e. $Q\eta = \lambda\eta$. Then $P^{-1} Q P P^{-1} \eta = \lambda P^{-1} \eta$. Combining with (4), one obtains

$$\begin{pmatrix} \mathbf{0} & I_n \\ -\alpha I_n & -L_s \end{pmatrix} P^{-1} \eta = \lambda P^{-1} \eta.$$

Let $\eta = \binom{x_r}{y_r}$, where $x_r \in \mathbb{R}^n$, $y_r \in \mathbb{R}^n$. Immediately one obtains

$$\begin{pmatrix} \mathbf{0} & I_n \\ -\alpha I_n & -L_s \end{pmatrix} \begin{pmatrix} K^{-1} x_r \\ K^{-1} y_r \end{pmatrix} = \lambda \begin{pmatrix} K^{-1} x_r \\ K^{-1} y_r \end{pmatrix},$$

i.e.

$$\begin{aligned} K^{-1} y_r &= \lambda K^{-1} x_r, \\ -\alpha K^{-1} x_r - L_s K^{-1} y_r &= \lambda K^{-1} y_r. \end{aligned} \tag{5}$$

Therefore,

$$-\alpha K^{-1} x_r - \lambda L_s K^{-1} x_r = \lambda^2 K^{-1} x_r. \tag{6}$$

Clearly, $\lambda \neq 0$ (otherwise, assume $\lambda = 0$, then by (5), one has $x_r = y_r = 0$, and hence $\eta = 0$. This contradicts the fact that η is an eigenvector associated with λ). Hence, by (6), one has $L_s K^{-1} x_r = -\frac{\lambda^2+\alpha}{\lambda} K^{-1} x_r$, i.e. $-\frac{\lambda^2+\alpha}{\lambda}$ is an eigenvalue of L_s and $K^{-1} x_r$ is its right eigenvector. Let $\xi = -\frac{\lambda^2+\alpha}{\lambda}$, $v_r = K^{-1} x_r$, then ξ is an eigenvalue of L_s, v_r is a right eigenvector associated with ξ. It is easy to know that $\lambda^2 + \lambda\xi + \alpha = 0$ and $x_r = K v_r$, $y_r = \lambda K v_r$. Thus $\eta = P(v_r^T, \lambda v_r^T)^T$.

Similarly, assume φ is a left eigenvector associated with λ, i.e. $\varphi^T Q = \lambda \varphi^T$, then $\varphi^T P P^{-1} Q P = \lambda \varphi^T P$. Let $\varphi = (\varphi_*^T, \varphi_o^T)^T$, where $\varphi_* \in \mathbb{R}^n$ and $\varphi_o \in \mathbb{R}^n$. Combining with (4), by direct calculations, one immediately gets $\varphi_*^T K L_s = -\frac{\lambda^2+\alpha}{\lambda} \varphi_*^T K = \xi \varphi_*^T K$. Therefore, $K^T \varphi_*$ is the left eigenvector associated with ξ. Let $v_l = K^T \varphi_*$, then $\varphi_* = (K^{-1})^T v_l$, $\varphi_o = -\frac{\lambda}{\alpha} (K^{-1})^T v_l$. Therefore, $\varphi = (P^{-1})^T (v_l^T, -\frac{\lambda}{\alpha} v_l^T)^T$. ■

Lemma 2.5: *Suppose $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}, \mathcal{D})$ and μ is an eigenvalue of Q. Then there exists eigenvalue γ of L, which satisfies $\mu^2 + \mu\gamma + \alpha = 0$, Furthermore, suppose ϕ and ζ are the left and right eigenvectors associated with μ, respectively. Then $\phi = (\beta_l^T, -\frac{\mu}{\alpha} \beta_l^T)^T$ and $\zeta = (\beta_r^T, \mu \beta_r^T)^T$, where β_l and β_r are the left and right eigenvectors associated with γ, respectively.*

Proof: It can be concluded by taking the similar procedures as in Lemma 2.4. ■

Remark 2.3: Compared with Lemma 2.4, where the relationship of eigenvalues between Q and L_s is shown under interactively balanced graph, here, in Lemma 2.5 the relationship of eigenvalues between Q and Laplacian L is elaborated, irrespective of the graph characteristic.

3. Main results

In this section, the weighted cooperative-competitive network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}, \mathcal{D})$ is first classified into three types, i.e. interactively balanced, interactively sub-balanced and interactively unbalanced, and then cluster consensus is dealt with for these three network types, respectively.

3.1 Interactively balanced network

Theorem 3.1: *The system in (1) achieves cluster consensus with protocol (2) if $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}, \mathcal{D})$ is interactively balanced and has a spanning tree. Moreover, as $t \to \infty$,*

$$\begin{aligned} r_i(t) &\to k_{1i}\Big(\cos(\sqrt{\alpha}t)\mathbf{w_l}^TK^{-1}r(0) \\ &\quad + \frac{1}{\sqrt{\alpha}}\sin(\sqrt{\alpha}t)\mathbf{w_l}^TK^{-1}v(0)\Big), \\ v_i(t) &\to k_{1i}\Big(-\sqrt{\alpha}\sin(\sqrt{\alpha}t)\mathbf{w_l}^TK^{-1}r(0) \\ &\quad + \cos(\sqrt{\alpha}t)\mathbf{w_l}^TK^{-1}v(0)\Big), \quad i = 1, 2, \ldots, n, \end{aligned} \tag{7}$$

where $K = \mathrm{diag}(k_{11}, k_{12}, \ldots, k_{1n})$ is defined as in Lemma 2.4, $\mathbf{w_l}$ is a left eigenvector of L_s associated with eigenvalue 0 and $\mathbf{w_l}^T\mathbf{1_n} = 1$.

Proof: Since $\mathcal{G}$ has a spanning tree, by Lemma 2.2, L_s has a simple eigenvalue 0 and all other eigenvalues have positive real parts. Without loss of generality, suppose $\xi_1 = 0, \xi_2, \ldots, \xi_n$ are n eigenvalues of L_s, it then readily follows that $\mathrm{Re}(\xi_i) > 0, i = 2, 3, \ldots, n$. According to Lemma 2.4, the eigenvalues of Q are

$$\begin{aligned} \lambda_{1+} &= \sqrt{\alpha}\mathbf{j}, \quad \lambda_{1-} = -\sqrt{\alpha}\mathbf{j}\ (\mathbf{j}^2 = -1); \\ \lambda_{i+} &= \frac{-\xi_i + \sqrt{\xi_i^2 - 4\alpha}}{2}, \\ \lambda_{i-} &= \frac{-\xi_i - \sqrt{\xi_i^2 - 4\alpha}}{2}, i = 2, 3, \ldots, n. \end{aligned}$$

Then, there exists an invertible matrix R such that

$$R^{-1}QR = \begin{pmatrix} \sqrt{\alpha}\mathbf{j} & 0 & \mathbf{0} \\ 0 & -\sqrt{\alpha}\mathbf{j} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & J \end{pmatrix}, \tag{8}$$

where J is the Jordan block with $\lambda_{i\pm}, (i = 2, 3, \ldots, n)$ on its diagonal. Obviously, $\mathrm{Re}(\lambda_{i+}) < 0, \mathrm{Re}(\lambda_{i-}) < 0, i = 2, 3, \ldots, n$.

Suppose l_1^T, l_2^T are the first and second rows of R^{-1}; $r_1, r_2 \in \mathbb{R}^{2n}$ are the first and second columns of R, respectively. Then, by (8), one immediately knows that l_1 and r_1 are left and right eigenvectors of Q associated with eigenvalue $\lambda_{1+} = \sqrt{\alpha}\mathbf{j}$; l_2 and r_2 are left and right eigenvectors of Q associated with eigenvalue $\lambda_{1-} = -\sqrt{\alpha}\mathbf{j}$, respectively. In addition, by definition, $L_s\mathbf{1_n} = 0$. Obviously, there exists $\mathbf{w_l}$ such that $\mathbf{w_l}^TL_s = 0$ and $\mathbf{w_l}^T\mathbf{1_n} = 1$. Therefore, by Lemma 2.4, one can derive that

$$\varphi_{1+} = \mathrm{diag}((K^{-1})^T, (K^{-1})^T)\left(\mathbf{w_l}^T, \frac{1}{\sqrt{\alpha}\mathbf{j}}\mathbf{w_l}^T\right)^T,$$

$$\eta_{1+} = \mathrm{diag}\,(K, K)\left(\mathbf{1_n}^T, \sqrt{\alpha}\mathbf{j}\mathbf{1_n}^T\right)^T;$$

$$\varphi_{1-} = \mathrm{diag}((K^{-1})^T, (K^{-1})^T)\left(\mathbf{w_l}^T, -\frac{1}{\sqrt{\alpha}\mathbf{j}}\mathbf{w_l}^T\right)^T,$$

$$\eta_{1-} = \mathrm{diag}\,(K, K)\left(\mathbf{1_n}^T, -\sqrt{\alpha}\mathbf{j}\mathbf{1_n}^T\right)^T,$$

are left and right eigenvectors associated with λ_{1+} and λ_{1-}, respectively. Note that both λ_{1+} and λ_{1-} are simple eigenvalues of Q, and hence their characteristic subspaces are 1-dimensional, say, $l_1 = \frac{1}{2}\varphi_{1+}, r_1 = \eta_{1+}; l_2 = \frac{1}{2}\varphi_{1-}, r_2 = \eta_{1-}$, since $R^{-1}R = I_{2n}$ and $l_1^Tr_1 = l_2^Tr_2 = 1$. By direct calculation, one gets

$$\begin{aligned} \lim_{t\to\infty} e^{Qt} &= R\begin{pmatrix} e^{\sqrt{\alpha}t\mathbf{j}} & 0 & \mathbf{0} \\ 0 & e^{-\sqrt{\alpha}t\mathbf{j}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} R^{-1} \\ &= e^{\sqrt{\alpha}t\mathbf{j}}r_1l_1^T + e^{-\sqrt{\alpha}t\mathbf{j}}r_2l_2^T, \end{aligned}$$

i.e.

$$\lim_{t\to\infty} e^{Qt} = \begin{pmatrix} \cos(\sqrt{\alpha}t)K\mathbf{1_n}\mathbf{w_l}^TK^{-1} & \frac{1}{\sqrt{\alpha}}\sin(\sqrt{\alpha}t)K\mathbf{1_n}\mathbf{w_l}^TK^{-1} \\ -\sqrt{\alpha}\sin(\sqrt{\alpha}t)K\mathbf{1_n}\mathbf{w_l}^TK^{-1} & \cos(\sqrt{\alpha}t)K\mathbf{1_n}\mathbf{w_l}^TK^{-1} \end{pmatrix}.$$

This together with (3) gives

$$\begin{aligned} r(t) &\to \Big(\cos(\sqrt{\alpha}t)\mathbf{w_l}^TK^{-1}r(0) \\ &\quad + \frac{1}{\sqrt{\alpha}}\sin(\sqrt{\alpha}t)\mathbf{w_l}^TK^{-1}v(0)\Big)K\mathbf{1_n}, \\ v(t) &\to \Big(-\sqrt{\alpha}\sin(\sqrt{\alpha}t)\mathbf{w_l}^TK^{-1}r(0) \\ &\quad + \cos(\sqrt{\alpha}t)\mathbf{w_l}^TK^{-1}v(0)\Big)K\mathbf{1_n}, t \to \infty. \end{aligned}$$

By Lemma 2.4, $K = \mathrm{diag}(k_{11}, k_{12}, \ldots, k_{1n})$ and $K\mathbf{1_n} = (k_{11}, k_{12}, \ldots, k_{1n})^T$. Thus, (7) holds. Take $c_{ij} = \frac{k_{1j}}{k_{1i}}, i, j = 1, 2, \ldots, n$. It is obvious that $k_{1i} \neq 0 (i = 1, 2, \ldots, n)$ only depends on interaction topology $\mathcal{G}$. Therefore, c_{ij} is determined by $\mathcal{G}$. According to Definition 2.3, cluster consensus is achieved. ■

Remark 3.1: (i) Theorem 3.1 gives the sufficient conditions for ensuring cluster consensus under interactively balanced network. It can be seen that states of all oscillators reach periodic orbits with the same frequency $\sqrt{\alpha}/2\pi$ regardless of cluster partitions.

(ii) From Theorem 3.1, one can see that if $k_{1i} = 1$, $\forall i = 1, 2, \ldots, n$, then $c_{ij} = \frac{k_{1j}}{k_{1i}} = 1, \forall i, j = 1, 2, \ldots, n$, i.e. all harmonic oscillators finally converge to the same cluster. In this situation, traditional consensus is achieved, and therefore cluster consensus is reduced to consensus in Ren (2008). If $k_{1i} = \pm 1$, then $c_{ij} = \pm 1$, $i, j = 1, 2, \ldots, n$. This implies that harmonic oscillators are separated into two clusters, and oscillators in different clusters reach an agreement whose values are the same in modulus but opposite in sign. Under this circumstance, cluster consensus degenerates to bipartite consensus in Liu et al. (2018).

3.2 Interactively sub-balanced network

Theorem 3.2: *The system in (1) with protocol (2) achieves cluster consensus if $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A},\mathcal{D})$ is interactively sub-balanced and has a spanning tree.*

Proof: since $\mathcal{G}$ is interactively sub-balanced, it does not contain directed cycles. This means that the nodes in $\mathcal{G}$ can be relabelled in a way that $i<j$ for $(i,j)\in\mathcal{E}$ and Laplacian L is lower triangular (Qin & Yu, 2013). This, together with the fact that $\mathcal{G}$ has a spanning tree, implies that L has exactly one zero eigenvalue and other ones are positive. Without loss of generality, suppose $\gamma_1=0,\gamma_2>0,\ldots,\gamma_n>0$ are eigenvalues of L. Then, by Lemma 2.5, the eigenvalues of Q are

$$\mu_{1+}=\sqrt{\alpha}\mathbf{j},\quad \mu_{1-}=-\sqrt{\alpha}\mathbf{j},\ (\mathbf{j}^2=-1)$$

$$\mu_{i+}=\frac{-\gamma_i+\sqrt{\gamma_i^2-4\alpha}}{2}, \tag{1}$$

$$\mu_{i-}=\frac{-\gamma_i-\sqrt{\gamma_i^2-4\alpha}}{2},\ i=2,3,\ldots,n. \tag{9}$$

Therefore, there exists an invertible matrix S such that

$$S^{-1}QS=\begin{pmatrix}\sqrt{\alpha}\mathbf{j} & 0 & \mathbf{0}\\ 0 & -\sqrt{\alpha}\mathbf{j} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & J_1\end{pmatrix}, \tag{10}$$

where J_1 is a Jordan block with $\mu_{i\pm}, i=2,3,\ldots,n$ on its diagonal. Obviously, $\mathrm{Re}(\mu_{i+})<0, \mathrm{Re}(\mu_{i-})<0, i=2,3,\ldots,n$. Suppose that s_1,s_2 are the first and second columns of matrix S, t_1^T,t_2^T are the first and second rows of matrix S^{-1}, respectively. Then t_1 and s_1 are the left and right eigenvectors of Q associated with eigenvalue $\mu_{1+}=\sqrt{\alpha}\mathbf{j}$, t_2 and s_2 are left and right eigenvectors of Q associated with eigenvalue $\mu_{1-}=-\sqrt{\alpha}\mathbf{j}$, respectively. In addition, by Lemma 2.5,

$$\phi_{1+}=\left(\beta_{l1}^T,\ \frac{1}{\sqrt{\alpha}\mathbf{j}}\beta_{l1}^T\right)^T,\quad \zeta_{1+}=\left(\beta_{r1}^T,\ \sqrt{\alpha}\mathbf{j}\beta_{r1}^T\right)^T,$$

$$\phi_{1-}=\left(\beta_{l1}^T,\ -\frac{1}{\sqrt{\alpha}\mathbf{j}}\beta_{l1}^T\right)^T,\quad \zeta_{1-}=\left(\beta_{r1}^T,\ -\sqrt{\alpha}\mathbf{j}\beta_{r1}^T\right)^T,$$

are, respectively, the left and right eigenvectors associated with μ_{1+} and μ_{1-}, where β_{l1} and β_{r1} are the left and right eigenvectors associated with $\gamma_1=0$, and $\beta_{l1}^T\beta_{r1}=1$. Notice that μ_{1+} and μ_{1-} are simple eigenvalues and thus their characteristic subspaces are 1-dimensional. Without loss of generality, suppose $t_1=\frac{1}{2}\phi_{1+}, s_1=\zeta_{1+}, t_2=\frac{1}{2}\phi_{1-}$ and $s_2=\zeta_{1-}$. By the definition of matrix exponential function,

$$\lim_{t\to\infty}e^{Qt}=S\begin{pmatrix}e^{\sqrt{\alpha}t\mathbf{j}} & 0 & \mathbf{0}\\ 0 & e^{-\sqrt{\alpha}t\mathbf{j}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{pmatrix}S^{-1}$$
$$=e^{\sqrt{\alpha}t\mathbf{j}}s_1t_1^T+e^{-\sqrt{\alpha}t\mathbf{j}}s_2t_2^T.$$

This together with (3) yields that when $t\to\infty$,

$$r(t)\to\left(\cos(\sqrt{\alpha}t)\beta_{l1}^Tr(0)+\frac{1}{\sqrt{\alpha}}\sin(\sqrt{\alpha}t)\beta_{l1}^Tv(0)\right)\beta_{r1},$$

$$v(t)\to\left(-\sqrt{\alpha}\sin(\sqrt{\alpha}t)\beta_{l1}^Tr(0)+\cos(\sqrt{\alpha}t)\beta_{l1}^Tv(0)\right)\beta_{r1}.$$

Since β_{r1} is a right eigenvector associated with eigenvalue 0, $L\beta_{r1}=\mathbf{0}$. Let $\beta_{r1}=(\beta_{r1}^{(1)},\beta_{r1}^{(2)},\ldots,\beta_{r1}^{(n)})^T$. Recalling that L is lower triangular, it then follows that $\beta_{r1}^{(1)}\neq 0$ (if not, $\beta_{r1}^{(i)}=0$, $i=2,3,\ldots,n$, and thus $\beta_{r1}=\mathbf{0}$. This contradicts the fact that β_{r1} is an eigenvector). Without loss of generality, we choose $\beta_{r1}^{(1)}=1$. Then $\beta_{r1}^{(j)}=\sum_{k\in\mathcal{N}_j}a_{jk}d_{jk}\beta_{r1}^{(k)}/\sum_{k\in\mathcal{N}_j}a_{jk}, j=2,3,\ldots,n$. Obviously, $\beta_{r1}^{(i)}$ is determined by $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A},\mathcal{D})$. Then, by (7), cluster consensus is achieved. ∎

Remark 3.2: Theorem 3.2 shows that for the interactively sub-balanced network with a spanning tree, cluster consensus can be reached by employing the protocol in (2). Compared with the interactively balanced case in Theorem 3.1, where cluster partition is only determined by d_{ij}, here, it depends on both a_{ij} and d_{ij}.

3.3 Interactively unbalanced network

In order to achieve cluster consensus of MCHOs for interactively unbalanced $\mathcal{G}$, we introduce the following pinning control.

Theorem 3.3: *The system in (1) with protocol (2) and the introduced pinning control achieves cluster consensus if $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A},\mathcal{D})$ is interactively unbalanced and has a spanning tree.*

Proof: Since $\mathcal{G}$ has a spanning tree, let's set the spanning tree as $S_T=(\mathcal{V},\mathcal{E}_T)$ with $\mathcal{E}_T\subset\mathcal{E}$. Without loss of generality, rearrange the indices of all the nodes such that $i<j$ for $(i,j)\in\mathcal{E}_T$. Assume that $\mathcal{P}_{1i}^{\mathcal{E}_T}\subset S_T$ is a directed path from 1 to i. Then, in $\mathcal{E}_T$, $\mathcal{P}_{1i}^{\mathcal{E}_T}$, $\forall i\neq 1$ is the only path from 1 to i. Let p_{1i} be the weighted product of $\mathcal{P}_{1i}^{\mathcal{E}_T}\subset S_T$. If $\mathcal{G}$ is interactively unbalanced, by definition, $\mathcal{G}$ contains directed cycles and there exists at least one $C_i\neq 1$ in $\hat{\mathcal{G}}$. Thus, there exists $(j,i)\in\mathcal{E}\backslash\mathcal{E}_T$ such that $p_{1i}^{-1}d_{ij}p_{1j}\neq 1$. Define $\Delta_1=\{i:(j,i)\in\mathcal{E}\backslash\mathcal{E}_T, p_{1i}^{-1}d_{ij}p_{1j}\neq 1\}$. Then, we can divide oscillators in (1) into two categories based on whether they are in Δ_1, and introduce the pinning control to oscillators in Δ_1, i.e.

$$\dot{v}_i(t)=-\alpha r_i(t)+\sum_{j\in\mathcal{N}_i}a_{ij}[d_{ij}v_j(t)-v_i(t)]+u_i^*(t),\quad i\in\Delta_1;$$
$$\dot{v}_i(t)=-\alpha r_i(t)+\sum_{j\in\mathcal{N}_i}a_{ij}[d_{ij}v_j(t)-v_i(t)],\quad i\notin\Delta_1, \tag{11}$$

where $u_i^*(t)$ denotes the pinning control introduced to the ith oscillator.

From the definition of interactively unbalanced graph, it follows that $\mathcal{G}$ contains directed cycles and there exists at least one weight product $C_i\neq 1$ in $\hat{\mathcal{G}}$. Next, we will introduce the pinning control according to weight products of directed cycles in $\hat{\mathcal{G}}$.

Case I: all $C_i > 0$ in $\hat{\mathcal{G}}$. In this case, for $i \in \Delta_1$, we design the pinning controller as

$$u_i^*(t) = -k_i v_i(t), \quad i \in \Delta_1, \tag{12}$$

where $k_i = \sum_{j\in\mathcal{N}_i} \tilde{a}_{ij} - l_{ii}$, $\tilde{a}_{ij} = a_{ij}p_{1i}^{-1}d_{ij}p_{1j}$ if $(j,i) \in \mathcal{E}\backslash\mathcal{E}_T$; $\tilde{a}_{ij} = a_{ij}$, otherwise. Apply (12) to the system in (11), then

$$\begin{pmatrix} \dot{r}(t) \\ \dot{v}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{0} & I_n \\ -\alpha I_n & -\tilde{L} \end{pmatrix} \begin{pmatrix} r(t) \\ v(t) \end{pmatrix},$$

where $\tilde{L} = (\tilde{l}_{ij})$,

$$\tilde{l}_{ij} = \begin{cases} \sum_{j\in\mathcal{N}_i} a_{ij}|p_{1i}^{-1}d_{ij}p_{1j}|, & i = j, \\ -a_{ij}d_{ij}sgn(p_{1i}^{-1}d_{ij}p_{1j}), & i \neq j. \end{cases}$$

Let $H = \mathrm{diag}(p_{11}, p_{12}, \ldots, p_{1n})$ with $p_{11} = 1$. Then, $H^{-1}\tilde{L}H = \tilde{L}_s$, where $\tilde{L}_s$ is the standard Laplacian. Taking the same proof process as in Theorem 3.1, we obtain that oscillators in (1) achieve cluster consensus.

Case II: there exists at least one $C_i < 0$ in $\hat{\mathcal{G}}$. Define $\Delta_2 = \{j : (j,i) \in \mathcal{E}\backslash\mathcal{E}_T, p_{1i}^{-1}d_{ij}p_{1j} < 0\}$. For $i \in \Delta_1$, we design the pinning controller as

$$u_i^*(t) = -k_{ii}v_i(t) + \sum_{j\in\mathcal{N}_i\cap\Delta_2} k_{ij}v_j(t), \quad i \in \Delta_1, \tag{13}$$

where $k_{ii} = \sum_{j\in\mathcal{N}_i} |\tilde{a}_{ij}| - l_{ii}$, $k_{ij} = -a_{ij}d_{ij}[1 - \mathrm{sgn}(\tilde{a}_{ij})](i \neq j)$. Applying (13) to the system in (11), one has

$$\begin{pmatrix} \dot{r}(t) \\ \dot{v}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{0} & I_n \\ -\alpha I_n & -\tilde{L} \end{pmatrix} \begin{pmatrix} r(t) \\ v(t) \end{pmatrix}.$$

Repeating the same procedure as in Case I, we obtain that oscillators in (1) achieve cluster consensus. ■

Remark 3.3: For $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}, \mathcal{D})$ being interactively unbalanced and having a spanning tree, using protocol (2) only the oscillators in (1) can be either convergent to zero or divergent, but not necessarily achieving cluster consensus. The pinning control is hence vital in this case.

Till now, cluster consensus for MCHOs has been achieved under three different networks, i.e. interactively balanced, interactively sub-balanced and interactively unbalanced, respectively. Note that in the above three cases, $\hat{\mathcal{G}}$ has directed cycles. It would be interesting to investigate cluster consensus for oscillators under networks where $\hat{\mathcal{G}}$ does not have directed cycles. As a special case of this, we will consider the network where $\mathcal{G}$ is a spanning tree.

Corollary 3.4: *The system in (1) with protocol (2) can achieve cluster consensus if $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}, \mathcal{D})$ is a spanning tree.*

Proof: If $\mathcal{G}$ is a spanning tree, then nodes in $\mathcal{G}$ can be relabelled in a way that $i < j$ for $(i,j) \in \mathcal{E}$. Obviously, $\mathcal{G}$ does not contain directed cycles. Thus, Laplacian L is lower triangular. Moreover, L has exactly one zero eigenvalue and other ones are positive. By using the similar analysis after (9), one obtains cluster consensus. ■

Remark 3.4: Different from the three networks, i.e. interactively balanced, interactively sub-balanced and interactively unbalanced studied in Zhan and Li (2018), where $\hat{\mathcal{G}}$ has directed cycles, here a special case of the network that $\hat{\mathcal{G}}$ does not have directed cycles is considered.

4. Simulations

In this section, numerical examples are given to validate the proposed theoretical results for cluster consensus.

Example 4.1: Consider a coupled harmonic oscillators system composed of five harmonic oscillators with $\alpha = 1$ in (1). Communication interactions among them are expressed by $\mathcal{G}_1 = (\mathcal{V}, \mathcal{E}_1, \mathcal{A}_1, \mathcal{D}_1)$ with $\mathcal{A}_1 = (a_{ij})_{5\times5}$ and $\mathcal{D}_1 = (d_{ij})_{5\times5}$, where $a_{21} = a_{43} = 1$, $a_{24} = 4$, $a_{32} = a_{53} = 2$, $d_{21} = 0.5$, $d_{23} = -0.5$, $d_{32} = -2$ and $d_{43} = d_{53} = 1$. Since node 2, 3 and 4 form a directed cycle and $d_{32}d_{43}d_{24} = 1$, $\mathcal{G}_1$ is interactively balanced. By Figure 1, $\mathcal{G}_1$ has a spanning tree. Set $r(0) = (-6, 2, -3, 1, 5)^T$, $v(0) = (5, 0, -5, 2, -1)^T$. Then cluster consensus is achieved under protocol (2), as shown in Figure 2. The oscillators go into three clusters, i.e. 1; 2; 3,4,5, and the state proportion of three clusters is 2: 1: -2. In light of Theorem 3.1, $K = \mathrm{diag}(k_{11}, k_{12}, k_{13}, k_{14}, k_{15}) = \mathrm{diag}(1, \frac{1}{2}, -1, -1, -1)$, and $c_{34} = k_{14}/k_{13} = 1$, $c_{45} = k_{15}/k_{14} = 1$, $c_{12} = k_{12}/k_{11} = \frac{1}{2}$, $c_{23} = k_{13}/k_{12} = -2$, $c_{13} = k_{13}/k_{11} = -1$. By Definition 2.3, oscillators 3, 4, 5 constitute a cluster, oscillators 1 and 2, respectively, constitute a cluster. This is consistent with the simulation results.

Example 4.2: Suppose the communication interactions among the five oscillators in Example 4.1 are expressed by $\mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_2, \mathcal{A}_2, \mathcal{D}_2)$, with $\mathcal{A}_2 = (a_{ij})_{5\times5}$ and $\mathcal{D}_2 = (d_{ij})_{5\times5}$, where $a_{21} = 2$, $a_{32} = 1$, $a_{42} = 3$, $a_{43} = 4$, $a_{53} = 3$, $d_{21} = 2$, $d_{32} = 0.5$, $d_{42} = 1$, $d_{43} = -1$ and $d_{53} = -0.5$. By Figure 3, there are no directed cycles in $\mathcal{G}_2$, but in $\hat{\mathcal{G}}_2$ node 2, 3 and 4 form a directed cycle and $d_{32}d_{43}d_{24} = -0.5 \neq 1$. So $\mathcal{G}_2$ is interactively sub-balanced. Obviously, $\mathcal{G}_2$ has a spanning tree. Let $r(0) = (-3, 5, -1, 0, 1)^T$ and $v(0) = (0, 5, 3, -1, -5)^T$. Then cluster consensus is achieved under protocol (2), as shown in Figure 4.

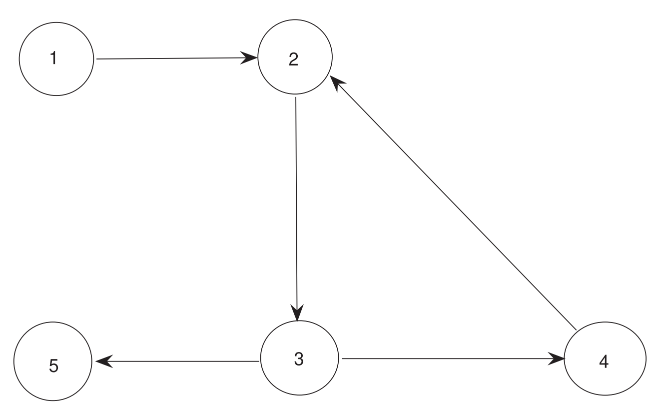

Figure 1. Interactively balanced topology graph $\mathcal{G}_1$.

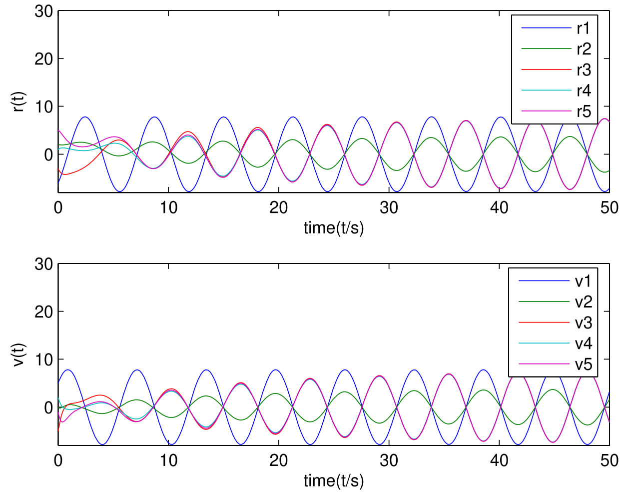

Figure 2. State trajectories of coupled harmonic oscillators in $\mathcal{G}_1$.

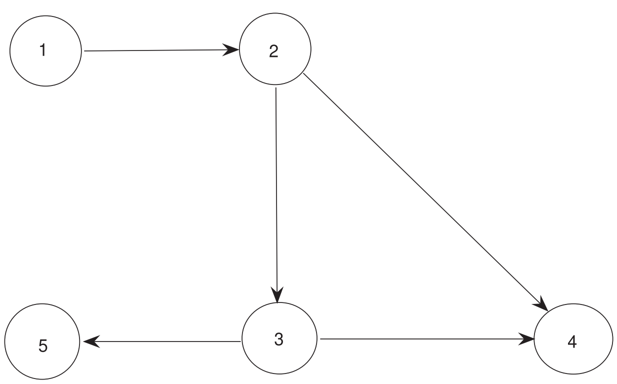

Figure 3. Interactively sub-balanced topology graph $\mathcal{G}_2$.

The oscillators go into four clusters, i.e. 1, 3; 2; 4; 5. In fact, by Theorem 3.2, we have $c_{13} = \beta_{r1}^3/\beta_{r1}^1 = 1$, $c_{23} = \beta_{r1}^3/\beta_{r1}^2 = \frac{1}{2}$, $c_{24} = \beta_{r1}^4/\beta_{r1}^2 = \frac{1}{7}$, $c_{25} = \beta_{r1}^5/\beta_{r1}^2 = -\frac{1}{4}$, $c_{34} = \beta_{r1}^4/\beta_{r1}^3 = \frac{2}{7}$, $c_{35} = \beta_{r1}^5/\beta_{r1}^3 = -\frac{1}{2}$, and $c_{45} = \beta_{r1}^5/\beta_{r1}^4 = -\frac{7}{4}$. i.e. oscillators 1 and 3 constitute a cluster, oscillators 2, 4 and 5 constitute a cluster, respectively. In particular, if the directed edge $(2, 4)$ in $\mathcal{G}_2$ is deleted, then $\mathcal{G}_2$ is a spanning tree. In this case cluster consensus is achieved under protocol (2), as shown in Figure 5. This validates the effectiveness of Corollary 3.4.

Example 4.3: Consider the five oscillators in Example 4.1 with $\mathcal{G}_3 = (\mathcal{V}, \mathcal{E}_3, \mathcal{A}_3, \mathcal{D}_3)$, as shown in Figure 6, where $\mathcal{A}_3 = (a_{ij}) \in \mathbb{R}^{5\times5}$, $a_{15} = 3$, $a_{21} = 2$, $a_{32} = 1$, $a_{43} = 4$, $a_{53} = 3$, and $\mathcal{D}_3 = (d_{ij}) \in \mathbb{R}^{5\times5}$, $d_{15} = -2$, $d_{21} = 2$, $d_{32} = 0.5$, $d_{43} = -1$, $d_{53} = -1$. By definition, $\mathcal{G}_3$ is interactively unbalanced, and has a spanning tree. To ensure cluster consensus, a pinning control is introduced as in Theorem 3.3. Clearly, $(5, 1) \in \mathcal{E}\backslash\mathcal{E}_T$ and $p_{11}^{-1} d_{15} p_{15} = 2 \neq 1$. Thus, node $1 \in \Delta_1$. By calculations, all weight products of directed cycles in $\hat{\mathcal{G}}_3$ are positive, and $d_{21}d_{32}d_{53}d_{15} = 2 \neq 1$. Then, by case I in Theorem 3.3, a pinning control $u_1^*(t) = -3v_1(t)$ is designed based on (12). Assume $r(0) = (-2, 1.5, -0.5, 1, 3)^T$ and $v(0) = (0, 1, 2, -1, -2)^T$. Then, cluster consensus is achieved as shown in Figure 7, where oscillators go into three clusters, i.e. 1, 3; 2; 4, 5.

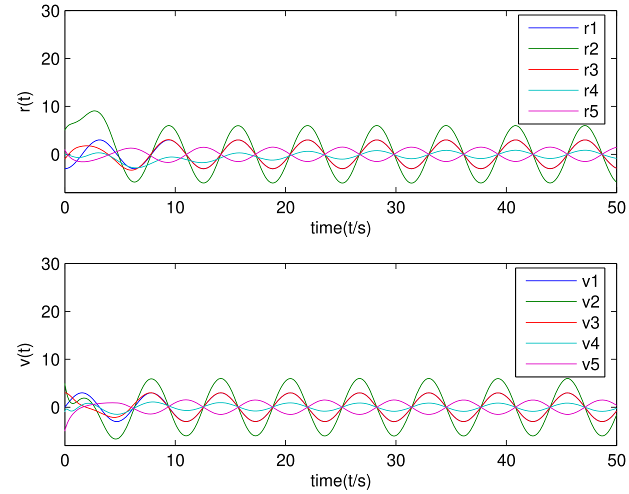

Figure 4. State trajectories of coupled harmonic oscillators in $\mathcal{G}_2$.

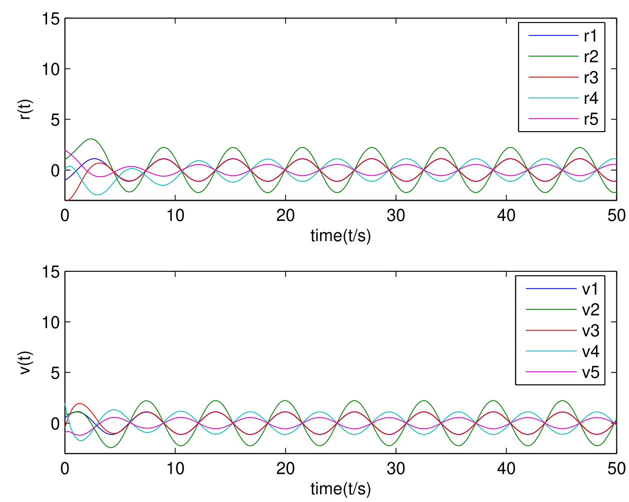

Figure 5. State trajectories of coupled harmonic oscillators under a spanning tree.

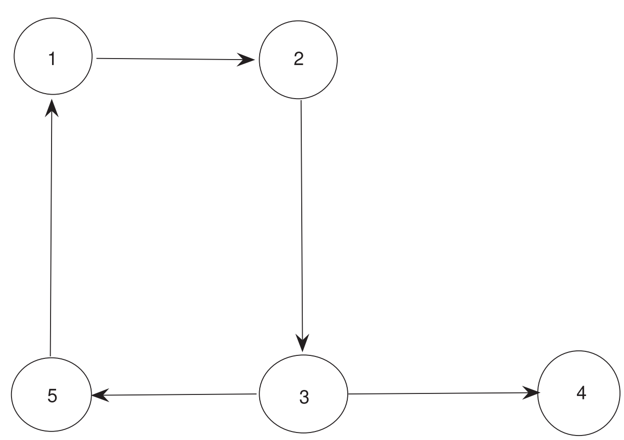

Figure 6. Interactively unbalanced topology graph $\mathcal{G}_3$.

Next, we change weights in $\mathcal{G}_3$ and obtain $\mathcal{G}_4 = (\mathcal{V}, \mathcal{E}_3, \mathcal{A}_3, \mathcal{D}_4)$, where $\mathcal{D}_4 = (d_{ij})$ and $d_{15} = 2$, $d_{21} = 2$, $d_{32} = 0.5$, $d_{43} = -1$, $d_{53} = -1$. $\mathcal{G}_4$ is also interactively unbalanced. By calculation, $(5, 1) \in \mathcal{E}\backslash\mathcal{E}_T$ and $p_{11}^{-1} d_{15} p_{15} = -2 < 0$, so node $5 \in \Delta_2$. Since $d_{21}d_{32}d_{53}d_{15} = -2 < 0$ in $\hat{\mathcal{G}}_4$, by case II

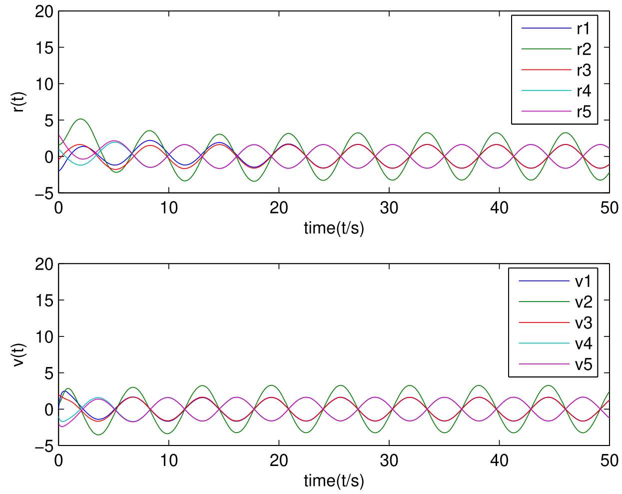

Figure 7. State trajectories with the pinning control in $\mathcal{G}_3$.

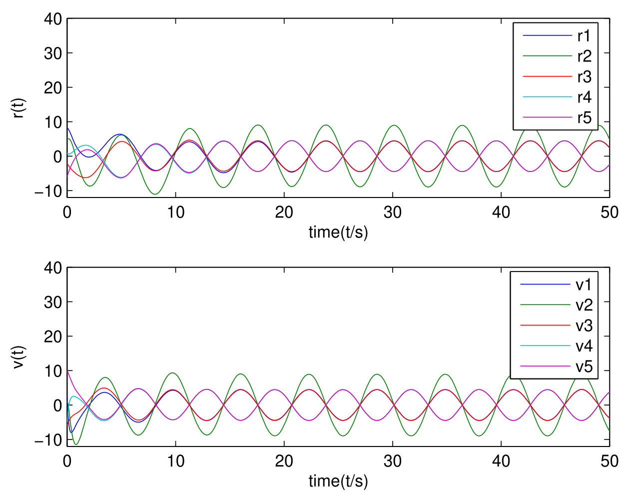

Figure 8. State trajectories with the pinning control in $\mathcal{G}_4$.

in Theorem 3.3, $u_1^*(t) = -k_{11}v_1(t) + k_{15}v_5(t) = -3v_1(t) - 12v_5(t)$. Cluster consensus is achieved as shown in Figure 8.

5. Concluding remarks

Cluster consensus of coupled harmonic oscillators is studied for weighted cooperative-competitive networks. Distributed control protocols are proposed for three kinds of weighted cooperative-competitive networks, respectively. Based on them, cluster consensus is achieved. Compared to conventional consensus and bipartite consensus in multi-agent systems, where only cooperative interactions and binary cooperative-competitive interactions are considered, respectively, here weighted cooperative-competitive interactions are taken into consideration, representing a more realistic model for social networks.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

The work was supported in part by the National Natural Science Foundation of China [grant number 61973183 and 62173317]; the National Key Research and Development Program of China [grant number 2018AAA0100801]; the Natural Science Foundation of Shandong Province [grant number ZR2019MF041]; and the Youth Creative Team Sci-Tech Program of Shandong Universities [grant number 2019KJI007].

ORCID

Yun-Bo Zhao http://orcid.org/0000-0002-3684-5297

References

Altafini, C. (2013). Consensus problems on networks with antagonistic interactions. *IEEE Transactions on Automatic Control*, *58*(4), 935–946. https://doi.org/10.1109/TAC.2012.2224251

Ballard, L., Cao, Y., & Ren, W. (2010). Distributed discrete-time coupled harmonic oscillators with application to synchronised motion coordination. *IET Control Theory and Applications*, *4*(5), 806–816. https://doi.org/10.1049/iet-cta.2009.0053

Belykh, V. N., Belykh, I. V., Hasler, M., & Nevidin, K. V. (2003). Cluster synchronization in three-dimensional lattices of diffusively coupled oscillators. *International Journal of Bifurcation and Chaos*, *13*(4), 755–779. https://doi.org/10.1142/S0218127403006923

Develer, Ü., & Akar, M. (2021). Cluster consensus in first and second-order continuous-time networks with input and communication delays. *International Journal of Control*, *94*(4), 961–976. https://doi.org/10.1080/00207179.2019.1625446

Dolby, A. S., & Grubb, T. C. (1998). Benefits to satellite members in mixed specifies foraging groups: an experimental analysis. *Animal Behaviour*, *56*(2), 501–509. https://doi.org/10.1006/anbe.1998.0808

Feng, Y., Xu, S., & Zhang, B. (2014). Group consensus control for double-integrator dynamic multiagent systems with fixed communication topology. *International Journal of Robust and Nonlinear Control*, *24*(3), 532–547. https://doi.org/10.1002/rnc.v24.3

Ge, X., Han, Q. L., & Zhang, X. M. (2018). Achieving cluster formation of multi-agent systems under aperiodic sampling and communication delays. *IEEE Transactions on Industrial Electronics*, *65*(4), 3417–3426. https://doi.org/10.1109/TIE.2017.2752148

Guo, X., Liang, J., & Fardoun, H. M. (2020). New methods to realize the cluster consensus for multi-agent networks. *Asian Journal of Control*, *22*(6), 2549–2557. https://doi.org/10.1002/asjc.v22.6

Liu, J., Li, H., Ji, J., & Luo, J. (2020). Group-bipartite consensus in the networks with cooperative-competitive interactions. *IEEE Transactions on Circuits and Systems-II: Express Briefs*, *67*(12), 3292–3296. https://doi.org/10.1109/TCSII.8920

Liu, J., Li, H. Y., & Luo, J. (2018). Bipartite consensus control for coupled harmonic oscillators under a coopetitive network topology. *IEEE Access*, 6, 3706–3714. https://doi.org/10.1109/ACCESS.2018.2790970

Ma, C. Q., Li, T., & Zhang, J. F. (2010). Consensus control for leader-following multi-agent systems with measurement noises. *Journal of Systems Science and Complexity*, *23*(1), 35–49. https://doi.org/10.1007/s11424-010-9273-4

Ma, C. Q., & Xie, L. H. (2020). Necessary and sufficient conditions for leader-following bipartite consensus with measurement noise. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, *50*(5), 1976–1981. https://doi.org/10.1109/TSMC.6221021

Ma, C. Q., & Zhang, J. F. (2010). Necessary and sufficient conditions for consensusability of linear multi-agent systems. *IEEE Transactions on Automatic Control*, *55*(5), 1263–1268. https://doi.org/10.1109/TAC.2010.2042764

Olfati-Saber, R., & Murray, R. M. (2004). Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, *49*(9), 1520–1533. https://doi.org/10.1109/TAC.2004.834113

Prorok, A., Hsieh, M. A., & Kumar, V. (2017). The impact of diversity on optimal control policies for heterogeneous robot swarms. *IEEE Transactions on Robotics*, *33*(2), 346–358. https://doi.org/10.1109/TRO.2016.2631593

Qin, J., Fu, W., Gao, H., & Zheng, W. X. (2017). Distributed k-means algorithm and fuzzy c-means algorithm for sensor networks based on multiagent consensus theory. *IEEE Transactions on Cybernetics*, *47*(3), 772–783. https://doi.org/10.1109/TCYB.2016.2526683

Qin, J., & Yu, C. (2013). Cluster consensus control of generic linear multi-agent systems under directed topology with acyclic partition. *Automatica*, *49*(9), 2898–2905. https://doi.org/10.1016/j.automatica.2013.06.017

Ren, W. (2008). Synchronization of coupled harmonic oscillators with local interaction. *Automatica*, *44*(12), 3195–3200. https://doi.org/10.1016/j.automatica.2008.05.027

Ren, W., & Beard, R. M. (2005). Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, *50*(5), 655–661. https://doi.org/10.1109/TAC.2005.846556

Su, H., Chen, M. Z., Wang, X., Wang, H., & Valeyev, N. V. (2013). Adaptive cluster synchronization of coupled harmonic oscillators with multiple leaders. *IET Control Theory and Applications*, *7*(5), 765–772. https://doi.org/10.1049/iet-cta.2012.0910

Tuna, S. E. (2017). Synchronization of harmonic oscillators under restorative coupling with applications in electrical networks. *Automatica*, *75*, 236–243. https://doi.org/10.1016/j.automatica.2016.09.035

Wu, W., Zhou, W., & Chen, T. (2009). Cluster synchronization of linearly coupled complex networks under pinning control. *IEEE Transactions on Circuits and Systems: I-Regular Papers*, *56*(4), 829–839. https://doi.org/10.1109/TCSI.2008.2003373

Yu, J., & Wang, L. (2010). Group consensus in multi-agent systems with switching topologies and communication delays. *Systems and Control Letters*, *59*(6), 340–348. https://doi.org/10.1016/j.sysconle.2010.03.009

Zhan, J., & Li, X. (2018). Cluster consensus in networks of agents with weighted cooperative-competitive interactions. *IEEE Transactions on Circuits and Systerms-II: Express Briefs*, *65*(2), 241–245. https://doi.org/10.1109/TCSII.8920

Zhang, H., & Ji, J. (2018). Synchronization of coupled harmonic oscillators without velocity measurements. *Nonlinear Dynamics*, *91*(4), 2773–2788. https://doi.org/10.1007/s11071-017-4045-5

Zhang, H., Wu, Q., & Ji, J. (2018). Synchronization of discretely coupled harmonic oscillators using sampled position states only. *IEEE Transactions on Automatic Control*, *63*(11), 3994–3999. https://doi.org/10.1109/TAC.9

Zhao, L., Wu, Q., & Wang, R. (2019). Pinning cluster synchronization of coupled nonidentical harmonic oscillators under directed topology. *Asian Journal of Control*, *21*(2), 1009–1016. https://doi.org/10.1002/asjc.v21.2

Zhao, M., Peng, C., Han, Q. L., & Zhang, X. M. (2020). Cluster consensus of multiagent systems with weighted antagonistic interactions. *IEEE Transactions on Cybernetics*. https://doi.org/10.1109/TCYB.2020.2966083.

Zhao, Q., Zheng, Y., & Zhu, Y. (2020). Consensus of hybrid multi-agent systems with heterogeneous dynamics. *International Journal of Control*, *93*(12), 2848–2858. https://doi.org/10.1080/00207179.2019.1566642

Zheng, Y., & Wang, L. (2012). Distributed consensus of heterogeneous multiagent systems with fixed and switching topologies. *International Journal of Control*, *85*(12), 1967–1976. https://doi.org/10.1080/00207179.2012.713986

Zhou, J., Zhang, H., Xiang, L., & Wu, Q. (2012). Synchronization of coupled harmonic oscillators with local instantaneous interaction. *Automatica*, *48*(8), 1715–1721. https://doi.org/10.1016/j.automatica.2012.05.022